arXiv:gr-qc/9808064v2 8 Jun 1999

Quantization of generally covariant systems with extrinsic time

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Abstract

A generally covariant system can be deparametrized by means of an "extrinsic" time, provided that the metric has a conformal "temporal" Killing vector and the potential exhibits a suitable behavior with respect to it. The quantization of the system is performed by giving the well ordered constraint operators which satisfy the algebra. The searching of these operators is enlightened by the methods of the BRST formalism.

PACS numbers: 04.60.Ds, 11.30.Ly

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General relativity and quantum mechanics are the most important achievements of physics in this century. It seems essential to find a quantum theory of gravity by embracing both theories in a consistent one. However, despite the many efforts that have been made, that program has not been sucessfully completed [1].

One of the most difficult features is the problem of time [2]. In quantum mechanics, time is an absolute parameter; it is not on an equal footing with the other coordinates that turn out to be operators and observables. Instead, in general relativity "time" is merely an arbitrary label of a spatial hypersurface, and physically significant quantities are independent of those labels: they are invariant under diffeomorfisms. General relativity is an example of a parametrized system (a system whose action is invariant under change of the integrating parameter). One can obtain such a kind of system by starting from an action which does not possess reparametrization invariance, and raising the time to the rank of a dynamical variable. So the original degrees of freedom and the time are left as functions of some physically irrelevant parameter. Time can be varied independently of the other degrees of freedom when a constraint together with the respective Lagrange multiplier are added. In this process, one ends with a special feature: the Hamiltonian is constrained to vanish.

Most efforts directed to quantize general relativity (or some minisuperspace models) emphasize the analogy with the relativistic particle [3,4]. Actually, both systems have Hamiltonian constraints $\mathcal{H}$ that are hyperbolic on the momenta. If the role of the squared mass is played by a positive definite potential, then the analogy is complete in the sense that time is hidden in configuration space. In fact, the positive definite potential guarantees that the temporal component of the momentum is never null on the constraint hypersurface. Thus the Poisson bracket $\{q^o, \mathcal{H}\}$ is also never null, telling us that q^o evolves monotonically on any dynamical trajectory; this is the essential property of time. In this case, Ref. [3] shows the consistent operator ordering obtained from the Becchi-Rouet-Stora-Tyutin (BRST) formalism.

Unfortunately that analogy cannot be considered too seriously because the potential in general relativity is the (non positive definite) spatial curvature. This means that the time

in general relativity must be suggested by another mechanical simile.

In order to essay a better mechanical model for general relativity, let us start with a system of n genuine degrees of freedom with a Hamiltonian $h = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu + v(q^\mu)$ and definite positive metric $g^{\mu\nu}$.

The dynamics of the system will not change if a function of t, namely $-\frac{t^2}{2}$ is added to the Hamiltonian. So we write the action

$$S = \int \left[p_\mu \frac{dq^\mu}{dt} - h(q^\mu, p_\mu) + \frac{t}{2}^2 \right] dt, \qquad \mu = 1, ..., n \tag{1}$$

The system is parametrized by regarding the integration variable t as a canonical variable whose conjugated momentum is (minus) the hamiltonian. This last condition enters the action as a constraint $\mathcal{H} = p_t + h - \frac{t}{2}^2$, and the action reads

$$S[q^\mu, p_\mu, t, p_t, N] = \int \left[p_t \frac{dt}{d\tau} + p_\mu \frac{dq^\mu}{d\tau} - N \left(p_t + h(q^\mu, p_\mu) - \frac{t}{2}^2 \right) \right] d\tau \tag{2}$$

where N is the Lagrange multiplier.

So far the constraint is parabolic in the momenta. However one can perform the canonical transformation,

$$q^0 = p_t, \qquad\qquad p_0 = -t \tag{3}$$

that turns the constraint $\mathcal{H}$ into a hyperbolic function of the momenta:

$$\mathcal{H} = q^0 + h - \frac{1}{2}p_0^2 = -\frac{1}{2}p_0^2 + \frac{1}{2}g^{\mu\nu}p_\mu p_\nu + v(q^\mu) + q^0 = \frac{1}{2}\mathcal{G}^{rs}p_r p_s + \mathcal{V}(q^r), \tag{4}$$

with $r, s = 0, 1, ..., n$. The metric components read $\mathcal{G}^{00} = -1$, $\mathcal{G}^{0\nu} = 0$, $\mathcal{G}^{\mu\nu} = g^{\mu\nu}$ and the potential is $\mathcal{V}(q^i) = v(q^\mu) + q^0$. Then $\mathcal{G}^{rs}$ is a Lorentzian metric, as is the supermetric in the Arnowitt-Deser-Misner (ADM) formalism of general relativity. The constraint (4) describes a parametrized system with extrinsic (hidden in the phase space) time [5], whose po tential is not positive definite.

For a complete analogy with general relativity, the "supermomenta constraints" can be introduced by adding m degrees of freedom q^a. Their spurious character is stated by m

linear and homogeneus constraints $G_a \equiv \xi_a^r p_r$, where $\vec{\xi_a}$ are m vectors fields tangent to the coordinate curves associated with the q^a's. These m constraints G_a can still be linearly combined

$$G_a \to G_{a'} = A_{a'}^{a}(q)\; G_a, \qquad\qquad \det A \neq 0, \tag{5}$$

to get an equivalent set of linear and homogeneous supermomenta constraints. The set $(\mathcal{H}, G_a)$ is first class.

Finally the dynamics of the system is obtained by varying the action

$$S[q^i, p_i, N, N^a] = \int \left[p_i \frac{dq^i}{d\tau} - N\mathcal{H} - N^a G_a \right] d\tau, \qquad i = 0, 1, ..., n+m \tag{6}$$

where N^a are the Lagrangian multipliers corresponding to the constraints G_a.

Such a system satisfies the following conditions [which can be read from the Hamiltonian constraint, Eq. (4)]:

$$\frac{\partial \mathcal{G}^{ij}}{\partial q^0} \approx 0 \tag{7}$$

$$\frac{\partial \mathcal{V}}{\partial q^0} = 1 \tag{8}$$

The symbol "$\approx$" means "weakly equal" (the equality is restricted to the submanifold defined by the constraints $G_a \approx 0$) and it replaces the ordinary equality because the metric has a nonphysical sector which may depend on q^0.

The parametrization of the system, which is still visible in Eq. (4) due to the special form of the potential and the components of the metric, can be masked by means of a general coordinate transformation. However, the distinctive geometrical properties of the system, namely Eqs. (7),(8), can be written in a geometrical (i.e., coordinate independent) language, by using Lie derivatives. Thus, Eqs. (4) and (7),(8) tell us that there exists a weakly *unitary* temporal Killing vector field satisfying

$$\mathcal{L}_{\vec{\xi}_0} \mathcal{G} \approx 0, \tag{9}$$

such that

$$\mathcal{L}_{\vec{\xi}_0}\mathcal{V} = 1. \tag{10}$$

Differing from those approaches where a hyperbolic constraint like Eq. (4) is compared with the one of a relativistic particle, and the parameter of the Killing vector is regarded as the time [6], in our treatment the time ($t = -p_0$) is the dynamical variable *conjugated* to the parameter of the Killing vector.

In order to quantize the theory, we must find well ordered first class constraint operators satisfying the quantum constraint algebra,

$$[\hat{\mathcal{H}}, \hat{G}_a] = \hat{c}^0_{0a}\hat{\mathcal{H}} + \hat{c}^b_{0a}\hat{G}_b \tag{11}$$

$$[\hat{G}_a, \hat{G}_b] = \hat{C}^c_{ab}(q)\hat{G}_c, \tag{12}$$

where the structure function c^b_{0a} is linear in the momenta, $c^b_{0a}(q,p) = c^{bj}_{0a}(q)p_j$.

However, it is apparent that the potential $\mathcal{V}$ commute with the linear constraints G_a (it is gauge invariant) and as a consecuence the structure function $\hat{c}^0_{0a}$ must vanish. The algebra (11),(12) with $\hat{c}^0_{0a} = 0$ was already solved in Ref. [3]. There, the Dirac constraint operators were obtained within the framework of the BRST formalism:

$$\hat{\mathcal{H}} = \frac{1}{2}f^{-\frac{1}{2}}\hat{p}_i\mathcal{G}^{ij}f\hat{p}_j f^{-\frac{1}{2}} + \frac{i}{2}f^{\frac{1}{2}}c^{aj}_{oa}\hat{p}_j f^{-\frac{1}{2}} + \mathcal{V} \tag{13}$$

and

$$\hat{G}_a = f^{\frac{1}{2}}\xi^i_a\hat{p}_i f^{-\frac{1}{2}}, \tag{14}$$

where the function $f = f(q)$ satisfies

$$C^b_{ab} = f^{-1}(f\xi^i_a)_{,i} = \mathrm{div}_{\tilde{\alpha}}\ \vec{\xi}_a, \tag{15}$$

($\tilde{\alpha}$ is the volume $\tilde{\alpha} \equiv f\ dq^0 \wedge ... \wedge dq^{n+m}$). [1]

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[1]The $(n+m+1)$-form $\tilde{\alpha}$ solving Eq. (15) is a volume in the configuration space $\mathcal{M}$: $\tilde{\alpha} \equiv$

The Dirac constraint operators (13),(14) were obtained from the quantum BRST generator, the central object of the method. The BRST generator is a fermionic real function in an extended phase space spanned by the original canonical pairs (q^i, p_i) and by $m+1$ fermionic canonical pairs $(\eta^a, \mathcal{P}_a)$ (one for each constraint). The quantum BRST generator is a nilpotent Hermitian operator which reads for the system under consideration:

$$\begin{aligned}\hat{\Omega} &= \hat{\eta}^o \hat{\mathcal{H}} + \hat{\eta}^a \hat{G}_a + \frac{1}{2}\hat{\eta}^o \hat{\eta}^a \hat{c}^b_{oa} \hat{\mathcal{P}}_b + \frac{1}{2}\hat{\eta}^a \hat{\eta}^b \hat{C}^c_{ab} \hat{\mathcal{P}}_c \\ &= \hat{\eta}^o \left(\frac{1}{2} f^{-\frac{1}{2}} \hat{p}_i \mathcal{G}^{ij} f \hat{p}_j f^{-\frac{1}{2}} + \frac{i}{2} f^{\frac{1}{2}} c^{aj}_{oa} \hat{p}_j f^{-\frac{1}{2}} + \mathcal{V} \right) + \hat{\eta}^a f^{\frac{1}{2}} \xi^i_a \hat{p}_i f^{-\frac{1}{2}} \\ &\quad + \frac{1}{2}\hat{\eta}^o \hat{\eta}^a (f^{\frac{1}{2}} c^{bj}_{oa} \hat{p}_j f^{-\frac{1}{2}} + f^{-\frac{1}{2}} \hat{p}_j c^{bj}_{oa} f^{\frac{1}{2}}) \hat{\mathcal{P}}_b + \frac{1}{2}\hat{\eta}^a \hat{\eta}^b C^c_{ab} \hat{\mathcal{P}}_c \end{aligned} \qquad (16)$$

(in this $\eta - \mathcal{P}$ ordering, the Dirac constraint operators and the structure function operators can be directly read from it [7]).

In Ref. [3] we started with a pseudo-Riemannian metric and a constant potential (a relativistic particle in curved space). That system had the property $c^0_{0a} = 0$, which facilitated the search for the nilpotent BRST generator. After that, a general (but positive definite) potential was introduced by means of a unitary transformation of the BRST generator, and c^0_{0a} turned to be non null. This procedure gave to the constraint operators the invariance under scaling of the superhamiltonian constraint (without having recourse to a curvature term).

In the present case the system is not a relativistic particle: the potential is not constant (and neither is it positive definite), and the time is not hidden among the coordinates. However c^0_{0a} is still null due to the gauge invariance of the potential. Once again, the

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$\tilde{E}^1 \wedge ... \wedge \tilde{E}^m \wedge \tilde{\omega}$, where $\{\tilde{E}^a\}$ is the dual basis of $\{\vec{\xi}_a\}$ in $T^*_{||}\mathcal{M}$ (the "longitudinal" tangent space, and $\tilde{\omega} = \omega(y)\, dy^0 \wedge ... \wedge dy^n$ is a closed n form where the y^r's are $n+1$ functions which are left invariant by the gauge transformations generated by the linear constraints $(dy^r(\vec{\xi}_a) = 0, \quad \forall r, a)$. $\tilde{\alpha}$ is the volume induced by the constraints in the gauge orbit, times a (nonchosen) volume in the "reduced" space. For a detailed demonstration, see Ref. [3].

invariance under scaling of the superhamiltonian will be introduced by performing a unitary transformation in the extended space.

The scaling of the Hamiltonian constraint

$$\mathcal{H} \to H = F\ \mathcal{H}, \qquad\qquad F > 0 \tag{17}$$

(then $\mathcal{G}^{ij} \to G^{ij} = F\ \mathcal{G}^{ij},\ \mathcal{V} \to V = F\ \mathcal{V}$) relaxes the geometrical properties of $\vec{\xi}_0$:

$$|\vec{\xi}_0| = 1 \quad\to\quad |\vec{\xi}_0| = F^{-\frac{1}{2}} \tag{18}$$

$$\mathcal{L}_{\vec{\xi}_0}\mathcal{G} \approx 0 \quad\to\quad \mathcal{L}_{\vec{\xi}_0}\mathbf{G} \approx C\ \mathbf{G} \tag{19}$$

$$\mathcal{L}_{\vec{\xi}_0}\mathcal{V} = 1 \quad\to\quad \mathcal{L}_{\vec{\xi}_0}V = C\ V + |\vec{\xi}_0|^{-2} \tag{20}$$

with $C(q) = \vec{\xi}_0(\ln F) = -2\vec{\xi}_0(\ln|\vec{\xi}_0|)$. Thus $\vec{\xi}_0$ becomes a weakly nonunitary conformal Killing vector.

At the quantum level, the corresponding scaling operation can be accomplished by performing the unitary transformation of the quantum BRST generator

$$\hat{\Omega} \to e^{i\hat{M}}\ \hat{\Omega}\ e^{-i\hat{M}}, \tag{21}$$

with

$$\hat{M} = [\hat{\mathcal{P}}_o\ \ln|\vec{\xi}_0|\ \hat{\eta}^o - \hat{\eta}^o\ \ln|\vec{\xi}_0|\ \hat{\mathcal{P}}_o], \qquad |\vec{\xi}_0| > 0 \tag{22}$$

leading to a new Hermitian and nilpotent BRST generator,

$$\begin{aligned}\hat{\Omega} = \hat{\eta}^o &\left(\frac{1}{2}|\vec{\xi}_0|^{-1}f^{-\frac{1}{2}}\hat{p}_i\mathcal{G}^{ij}f\hat{p}_jf^{-\frac{1}{2}}|\vec{\xi}_0|^{-1} + \frac{i}{2}|\vec{\xi}_0|^{-1}f^{\frac{1}{2}}c^{aj}_{oa}\hat{p}_jf^{-\frac{1}{2}}|\vec{\xi}_0|^{-1} + |\vec{\xi}_0|^{-2}\mathcal{V}\right)\\ &+\hat{\eta}^a|\vec{\xi}_0|f^{\frac{1}{2}}\xi^i_a\hat{p}_if^{-\frac{1}{2}}|\vec{\xi}_0|^{-1} - 2\hat{\eta}^o\hat{\eta}^a\xi^i_a(\ln|\vec{\xi}_0|)_{,i}\hat{\mathcal{P}}_o\\ &+\frac{1}{2}\hat{\eta}^o\hat{\eta}^a|\vec{\xi}_0|^{-1}(f^{\frac{1}{2}}c^{bj}_{oa}\hat{p}_jf^{-\frac{1}{2}} + f^{-\frac{1}{2}}\hat{p}_jc^{bj}_{oa}f^{\frac{1}{2}})|\vec{\xi}_0|^{-1}\hat{\mathcal{P}}_b + \frac{1}{2}\hat{\eta}^a\hat{\eta}^bC^c_{ab}\hat{\mathcal{P}}_c,\end{aligned} \tag{23}$$

which corresponds to constraint operators satisfying the scaling invariance (so that $G^{ij} = |\vec{\xi}_0|^{-2}\ \mathcal{G}^{ij}$, $V = |\vec{\xi}_0|^{-2}\ \mathcal{V}$, and $C^{bj}_{oa} = |\vec{\xi}_0|^{-2}\ c^{bj}_{oa}$)

$$\hat{H} = \frac{1}{2}|\vec{\xi}_0|^{-1}f^{-\frac{1}{2}}\hat{p}_i|\vec{\xi}_0|^2G^{ij}f\hat{p}_jf^{-\frac{1}{2}}|\vec{\xi}_0|^{-1} + \frac{i}{2}|\vec{\xi}_0|f^{\frac{1}{2}}C^{aj}_{oa}\hat{p}_jf^{-\frac{1}{2}}|\vec{\xi}_0|^{-1} + V, \tag{24}$$

$$\hat{G}_a = |\vec{\xi}_0| f^{\frac{1}{2}} \xi_a^i \hat{p}_i f^{-\frac{1}{2}} |\vec{\xi}_0|^{-1}, \tag{25}$$

with the corresponding set of structure functions,

$$\hat{C}_{oa}^o = -2\xi_a^i (\ln|\vec{\xi}_0|)_{,i}, \tag{26}$$

$$\hat{C}_{oa}^b = \frac{1}{2}\left(|\vec{\xi}_0| f^{\frac{1}{2}} C_{oa}^{bj} \hat{p}_j f^{-\frac{1}{2}} |\vec{\xi}_0|^{-1} + |\vec{\xi}_0|^{-1} f^{-\frac{1}{2}} \hat{p}_j C_{oa}^{bj} f^{\frac{1}{2}} |\vec{\xi}_0|\right), \tag{27}$$

$$\hat{C}_{ab}^c = C_{ab}^c, \tag{28}$$

all of them properly ordered for satisfying the constraint algebra,

$$[\hat{H}, \hat{G}_a] = \hat{C}_{oa}^o \hat{H} + \hat{C}_{oa}^b (q,p) \hat{G}_b, \tag{29}$$

$$[\hat{G}_a, \hat{G}_b] = \hat{C}_{ab}^c (q) \hat{G}_c. \tag{30}$$

The quantization procedure is not completed without a physical inner product where the spureous degree of freedom are frozen by means of gauge fixing conditions. In addition to the m gauge conditions χ^a related to spatial spureous degrees of freedom, one should deal with the reparametrization invariance, which is associated with the inclusion of time among the dynamical variables. This is an easy task, as long as one follows the paramerization process exposed at the very beginning of the present work. At the level of Eq. (2), it is apparent that one should insert a delta, $\delta(t-t_0)$ ($\{t-t_0, \mathcal{H}\} = 1$), to regularize the inner product, which means to take the inner product at a given time t_0,

$$(\varphi_1, \varphi_2)_{t_0} = \int dt dq \left[\prod^m \delta(\chi)\right] \; J \; \delta(t-t_0) \; \varphi_1^*(t, q^\gamma) \; \varphi_2(t, q^\gamma), \tag{31}$$

where $\gamma = 1, ..., n+m$ and J is the Faddeev-Popov determinant associated with the linear constraints. Then, through a canonical transformation, the time is associated with the momenta [Eq. (3)]; so by changing to this representation (by transforming Fourier the wave function) one obtains the inner product:

$$(\varphi_1, \varphi_2) = \frac{1}{2\pi} \int dq\ dq^0\ dq'^0\ \left[\prod^m \delta(\chi)\right]\ J\ e^{-it_0(q^0-q'^0)} \varphi_1^*(q^0, q^\gamma)\ \varphi_2(q'^0, q^\gamma). \tag{32}$$

When the Hamiltonian constraint is scaled [Eq. (17)], the physical inner product must remain invariant. On account of the behavior of the wave function under scaling, which is apparent in the structure of the constraint operators [see Eq. (34) below], the (originally unitary) norm of $\vec{\xi}_0$ must appear in the physical inner product

$$(\varphi_1, \varphi_2) = \frac{1}{2\pi} \int dq\ dq^0 \left[\prod^m \delta(\chi)\right]\ J\ \varphi_1^*(q^0, q^\gamma)\ |\vec{\xi}_0|^{-1} \int dq'^0\ e^{-it_0(q^0-q'^0)} |\vec{\xi}_0|^{-1} \varphi_2(q'^0, q^\gamma). \tag{33}$$

It should be noticed that the integration in the coordinate q'^0 is evaluated along the vector field lines of $\vec{\xi}_0$.

After completing the quantization, one can further understand the obtained ordering, Eqs. (24),(28). It fulfills the invariance properties imposed to the theory: (i) coordinate changes, (ii) combinations of the supermomenta [Eq. (5)], and (iii) scaling of the super-Hamiltonian [Eq. (17)]. The physical gauge-invariant inner product of the Dirac wave functions, Eq. (33), must be invariant under any of these transformations. On account of the change of the Faddeev-Popov determinant under (ii) and (iii), the inner product will remain invariant if the Dirac wave function changes according to

$$\varphi \to \varphi' = (\det A)^{\frac{1}{2}} |\vec{\xi}_0|\ \varphi. \tag{34}$$

So, the factors $f^{\pm\frac{1}{2}}$, $|\vec{\xi}_0|^{\pm 1}$ in the constraint operators are just what are needed in order that $\hat{G}_a \varphi$, $\hat{H}\varphi$, and $\hat{C}^b_{oa}\varphi$ transform as φ, so preserving the geometrical character of the Dirac wave function [3].

Concerning the extension of the here exposed treatment to general relativity, Kuchař has shown in Ref. [6] that a conformal timelike Killing vector actually exits in the superspace of the ADM formalism. But, the question whether or not it satisfies property (20) remains open.

As a final remark, it is worth mentioning that although the idea of an extrinsic time is not new in general relativity [8], its use in the quantization problems is rather scarce [9,10].

As it was shown, the mechanical model presented here can lead to a better understanding of its implementation.

ACKNOWLEDGMENTS

This research was supported by Universidad de Buenos Aires (Proy. TX 064) and Consejo Nacional de Investigaciones Científicas y Técnicas.

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